

1 Derivative at a point

If a function f is defined on a neighbourhood of a point c and $\lim_{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}$ or $\lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c}$ exists, then the function is said to be derivable at point $x = c$ and the derivative is denoted by $f'(c)$. $f'(c)$ is called the derivative of the function f at point $x = c$.

A function f is said to be derivable from the right or left at a point c if $\lim_{h \rightarrow 0^+} \frac{f(c+h)-f(c)}{h}$ or $\lim_{h \rightarrow 0^-} \frac{f(c+h)-f(c)}{h}$ exists finitely.

Example 1.1. Let $f(x) = |x| \quad \forall x \in \mathbb{R}$. Verify whether f is differentiable at $x = 0$.

Sol. Given function

$$\begin{aligned} f(x) &= |x| \\ &= x \quad x > 0 \\ &= -x \quad x < 0 \\ &= 0 \quad x = 0 \end{aligned}$$

By definition,

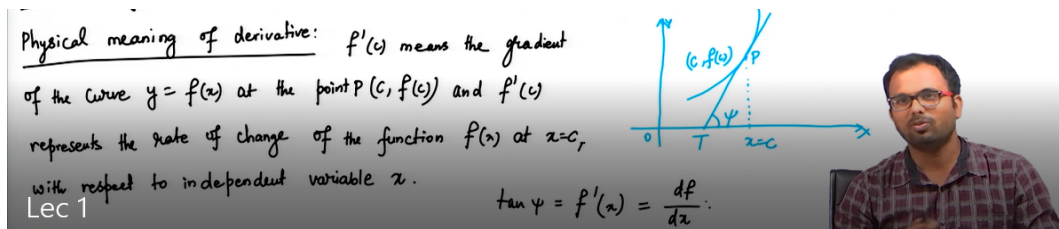
$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(c+h)-f(c)}{h} &= \lim_{h \rightarrow 0^+} \frac{h-0}{h} = 1 = Rf'(0) \\ \lim_{h \rightarrow 0^-} \frac{f(c+h)-f(c)}{h} &= \lim_{h \rightarrow 0^-} \frac{-h-0}{h} = -1 = Lf'(0) \end{aligned}$$

i.e, $Rf'(0) \neq Lf'(0) \Rightarrow f$ is not differentiable at $x = 0$. But, $|f(x) - f(0)| = ||x| - 0| = |x| < \varepsilon$ whenever $0 < |x| < \delta \Rightarrow f$ is continuous at 0.

Remark. Every differentiable function is continuous but converse may not be true.

Physical meaning of derivative: Derivative of a function f at a point $x = c$ i.e. $f'(c)$ means the gradient of the curve $y = f(x)$ at point $P(c, f(c))$ and $f'(c)$ represents the rate of change of the function $f(x)$ at $x = c$ with respect to the independent variable x .

$$\tan \psi = f'(x) = \frac{df}{dx}$$



Example 1.2. Let us consider the function

$$\begin{aligned} f(x) &= x \sin \frac{1}{x} \quad \text{if } x \neq 0 \\ &= 0 \quad \text{if } x = 0 \end{aligned}$$

Then, verify that $f(x)$ is differentiable at $x = 0$.

Sol. Here $|f(x) - f(0)| = |x \sin \frac{1}{x} - 0| = |x \sin \frac{1}{x}| = |x| |\sin \frac{1}{x}|$

$$\begin{aligned} |f(x) - f(0)| &= |x \sin \frac{1}{x} - 0| = |x \sin \frac{1}{x}| = |x| |\sin \frac{1}{x}| \\ \Rightarrow |f(x) - f(0)| &\leq |x| < \varepsilon \quad \text{whenever} \quad 0 < |x| < \delta \\ \Rightarrow f &\text{ is continuous at } x = 0. \end{aligned}$$

By definition,

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

But $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist. $f'(0) = \lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist. This implies that the function f is not differentiable at $x = 0$.

Example 1.3. Let us define

$$\begin{aligned} f(x) &= x^2 \sin \frac{1}{x} \quad \text{if } x \neq 0 \\ &= 0 \quad \text{if } x = 0 \end{aligned}$$

Sol. When $x \neq 0$, then $f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} - \frac{1}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

2 Mean Value Theorem

Theorem 2.1. (*Rolle's Theorem*) If a function f , defined on $[a, b]$, is

- i) continuous on $[a, b]$,
- ii) derivable on (a, b) ,
- iii) $f(a) = f(b)$.

Then, $\exists$ at least one real number $c \in (a, b)$ such that $f'(c) = 0$.

Example 2.1. Prove that between any two real roots of $e^x \sin x = 1$, there exists a real root or at least one real root of $e^x + 1 = 0$.

Sol: Let us consider $f(x) = e^{-x} - \sin x$. Let α and β be two real roots of $f(x) = 0$. Then, $f(\alpha) = f(\beta) = 0$. Both e^{-x} and $\sin x$ are continuous for all real x and derivable everywhere. f satisfies all the properties of *Rolle's Theorem* on $[\alpha, \beta]$, therefore $\exists$ a $\tau \in (\alpha, \beta)$ such that

$$\begin{aligned} f'(\tau) &= 0, \quad \alpha < \tau < \beta \\ \Rightarrow -e^{-\tau} - \cos \tau &= 0 \\ \Rightarrow e^{\tau} \cos \tau + 1 &= 0 \end{aligned}$$

Thus, there exists at least one real root $\tau \in (\alpha, \beta)$ of the equation $e^x \cos x + 1 = 0$.

Example 2.2. Verify Rolle's Theorem for $f(x) = x^3$ on $[-1, 1]$.

Sol. f is continuous on $[-1, 1]$. f is differentiable on $(-1, 1)$. $f(-1) \neq f(1) \Rightarrow$ The last criteria in Rolle's Theorem is not satisfied. Here $f'(x) = 3x^2 \forall x \in (-1, 1)$. Obviously $x = 0 \in (-1, 1)$ such that $f'(0) = 0$. This example shows that the condition in Rolle's Theorem is not necessary.

Theorem 2.2. (*Lagrange's Mean Value Theorem*) Let a real valued function f defined on $[a, b]$ be

- i) continuous on $[a, b]$,
- ii) differentiable in (a, b) .

Then there exists at least one point $c \in (a, b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

Example 2.3. Apply MVT to prove that

$$\frac{x}{1+x} < \log_e(1+x) < x, \quad \forall x > 0$$

Sol. Let $f(x) = \log_e(1+x)$. then the function f is continuous and differentiable on $[0, x]$.
By MVT

$$\begin{aligned} f(x) - f(0) &= xf'(\theta x), \quad \text{where } 0 < \theta < 1 \\ \Rightarrow \log(1+x) &= \frac{x}{1+\theta x} \end{aligned}$$

Now, $0 < \theta < 1$ and $x > 0$ then

$$\begin{aligned} 0 &< \theta x < x \\ \Rightarrow 1 &< 1 + \theta x < 1 + x \\ \Rightarrow \frac{1}{1+x} &< \frac{1}{1+\theta x} < 1 \\ \Rightarrow \frac{x}{1+x} &< \frac{x}{1+\theta x} < x \\ \Rightarrow \frac{x}{1+x} &< \log_e(1+x) < x \quad \forall x > 0 \end{aligned}$$

Alternative Statement. If a real valued function f is continuous in $[a, a+h]$, $h > 0$ and it is differentiable in $(a, a+h)$. Then there exists a proper fraction θ ($0 < \theta < 1$) such that $f(a+h) - f(a) = hf'(a+\theta h)$.

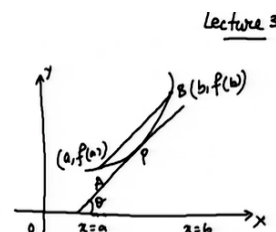
Example 2.4. By LMVT, prove that $0 < \frac{1}{\log_e(1+x)} - \frac{1}{x} < 1$.

Sol. Choose $f(x) = \log_e(1+x)$. Left as an exercise.

Geometrical Interpretation of LMVT. If a curve $y = f(x)$ is continuous from $x = a$ to $x = b$ and has a definite tangent at each point of the curve between $x = a$ to $x = b$, then geometrically the LMVT means that there is at least one point on the curve between $A(a, f(a))$ and $B(b, f(b))$ where the tangent is parallel to the chord AB .

§ Geometrical Interpretation of LMVT:

If a curve $y = f(x)$ is continuous from $x = a$ to $x = b$, and has a definite tangent at each point of the curve between $x = a$ to $x = b$, then geometrically the LMVT means that there is at least one point on the curve between $A(a, f(a))$ and $B(b, f(b))$ where the tangent is parallel to the chord AB .



Theorem 2.3. (Cauchy Mean Value Theorem) If two real valued functions f and g defined on a $[a, b]$ are such that

- i) f and g are continuous on $[a, b]$
 - ii) f and g are differentiable on (a, b) , i.e., f' and g' exists on (a, b)
 - iii) for no point in (a, b) such that $g'(x) = 0$, i.e., $g'(x) \neq 0 \quad \forall x \in (a, b)$.
- Then, there exists at least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Note: LMVT can be derive from Cauchy's MVT by taking $g(x) = x$ on $[a, b]$.

Example 2.5. If $f'(x)$ exists on $(0, 1)$, then show that by Cauchy's MVT that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one root in $(0, 1)$.

Sol. We consider $g(x) = x^2$ on $[0, 1]$. Then both $f(x)$ and $g(x)$ are continuous on $[0, 1]$ and they are differentiable on $(0, 1)$. Also, $g'(x) = 2x \neq 0 \forall x \in (0, 1)$. Then by Cauchy MVT, $\exists$ at least one point, say $c \in (0, 1)$, such that

$$\frac{f(1) - f(0)}{1 - 0} = \frac{f'(c)}{g'(c)} = \frac{f'(c)}{2c}$$

This implies that there exists at least one $c \in (0, 1)$ which satisfies $f(1) - f(0) = \frac{f'(x)}{2x}$.

Example 2.6. In Cauchy's MVT for the functions $\phi(x) = e^x$ and $f(x) = e^{-x}$, show that c is the arithmetic mean between a and b .

Sol. We are given that $\phi(x) = e^x$ and $f(x) = e^{-x}$. Both ϕ and f are continuous on $[a, b]$ and they are differentiable on (a, b) . Also, $f'(x) = -e^{-x} \neq 0 \forall x \in (a, b)$. The condition for CMVT are satisfied. Therefore there exists at least one $c \in (a, b)$ such that

$$\begin{aligned} \frac{\phi(a) - \phi(b)}{f(a) - f(b)} &= \frac{\phi'(c)}{f'(c)} \\ \Rightarrow \frac{e^b - e^a}{e^{-b} - e^{-a}} &= \frac{e^c}{-e^{-c}} = -e^{2c} \\ \Rightarrow \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} &= -e^{2c} \\ \Rightarrow -e^{a+b} &= -e^{2c} \\ \Rightarrow a + b &= 2c \\ \Rightarrow c &= \frac{a+b}{2} \end{aligned}$$

3 Higher Order Derivatives

$$\begin{aligned} y = f(x) &\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(f(x)) = y_1 = y' \\ \frac{d^2y}{dx^2} &= \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{df}{dx}\right) = y_2 = y'' \\ \frac{d^3y}{dx^3} &= \frac{d^2}{dx^2}\left(\frac{df}{dx}\right) = y_3 = y''' \\ &\vdots \\ \frac{d^ny}{dx^n} &= \frac{d^{n-1}}{dx^{n-1}}\left(\frac{df}{dx}\right) = y_n = y^n, \quad n \in \mathbb{N} \end{aligned}$$

Sometimes, $y_n = y^n$ is denoted by $y^{(n)}$ or D^n where $D = \frac{d}{dx}$.

Example 3.1. Let $y = f(x) = x^n$, $n \in \mathbb{N}$. Then find $D^m y = \frac{d^m y}{dx^m}$.

Sol.

$$\begin{aligned} Dy &= nx^{n-1} \\ D^2y &= d(dy) = n(n-1)x^{n-2} \\ &\vdots \\ D^m y &= n(n-1) \cdots (n-m+1)x^{n-m}, \end{aligned}$$

when $m = n$, then $D^m y = n(n-1) \cdots 1 x^0 = n!$

Example 3.2. Let $y = \log_e(ax + b)$, $a, b > 0$

Sol.

$$\begin{aligned} y_1 &= Dy = \frac{1}{ax+b}a = \frac{a}{ax+b} \\ y_2 &= -\frac{a^2}{(ax+b)^2} \\ &\vdots \\ y_n &= D^n y = d^{n-1} y_1 = ad^{n-1} \left(\frac{1}{ax+b} \right) \\ &= \frac{a^n (-1)^{n-1} (n-1)!}{(ax+b)^n} \\ \Rightarrow y_n &= \frac{a^n (-1)^{n-1} (n-1)!}{(ax+b)^n} \end{aligned}$$

Theorem 3.1. (Leibnitz Rule Differentiation for Product of Functions) If u and v be two functions of x defined on an interval $[a, b]$ and each is differentiable n times w.r.t. " x ", the n -th derivative of the product is given by

$$(uv)_n = D^n(uv) = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + \cdots + {}^n C_r u_{n-r} v_r + \cdots + u v_n$$

where the suffixes denote the order of derivative.

Example 3.3. If $y = \cos(m \sin^{-1} x)$, then show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$$

Sol.

$$\begin{aligned} y &= \cos(m \sin^{-1} x) \Rightarrow y_1 = Dy = -\sin(m \sin^{-1} x) \frac{m}{\sqrt{1-x^2}} \\ \Rightarrow (1-x^2)y_1^2 &= m^2 \sin^2(m \sin^{-1} x) = m^2 [1 - \cos^2(m \sin^{-1} x)] = m^2 (1-y^2) \\ &\text{(differentiating w.r.t. } x) \\ \Rightarrow (1-x^2)^2 y_1 y_2 - 2xy_1^2 &= -2m^2 y y_1 \\ \Rightarrow (1-x^2)y_2 - xy_1 + m^2 y &= 0 \end{aligned}$$

Differentiating w.r.t. x on both sides n times,

$$\begin{aligned} (1-x^2)y_{n+2} + {}^n C_1 y_{n+1}(-2x) + {}^n C_2 y_n(-2) + {}^n C_3 y_{n-1}(0) + \cdots - xy_{n+1} - {}^n C_1 y_n - \cdots + m^2 y_n &= 0 \\ \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n &= 0 \end{aligned}$$

Example 3.4. If $y = \frac{\log x}{x}$, then prove that $y_n = \frac{(-1)^n n!}{x^{n+1}} \left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} \right]$.

Sol. Let us take $u = \frac{1}{x}$ and $v = \log x$. Then

$$y_n = (uv)_n = u_n v + \sum_{r=1}^n {}^n C_r u_{n-r} v_r$$

Here, $u_n = \frac{(-1)^n n!}{x^{n+1}}$, $u_{n-r} = \frac{(-1)^{n-r} (n-r)!}{x^{n-r+1}}$ and $v_r = \frac{(-1)^{r-1} (r-1)!}{x^r}$ Now

$$\begin{aligned} y_n &= (uv)_n = u_n v + \sum_{r=1}^n {}^n C_r u_{n-r} v_r \\ &= u_n = \frac{(-1)^n n!}{x^{n+1}} \log x + \sum_{r=1}^n {}^n C_r \frac{(-1)^{n-r} (n-r)!}{x^{n-r+1}} \cdot \frac{(-1)^{r-1} (r-1)!}{x^r} \\ &= \frac{(-1)^n n!}{x^{n+1}} \left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} \right] \end{aligned}$$

Theorem 3.2. (Taylor's Theorem with General Form of Remainder) If a real valued function defined on $[a, b]$ or $[a, a + h]$ where $a + h = b$ be such that

i) f^{n-1} is continuous on $[a, a + h]$,

ii) f^n exists on $(a, a + h)$,

then there exists a positive proper fraction $\theta (0 < \theta < 1)$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n,$$

where $R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(a + \theta h)$, p being a positive integer $\leq n$.

Theorem 3.3. (Taylor's Theorem with Cauchy Form of Remainder) If a real valued function defined on $[a, b]$ or $[a, a + h]$ where $a + h = b$ be such that

i) f^{n-1} is continuous on $[a, a + h]$,

ii) f^n exists on $(a, a + h)$,

then there exists a positive proper fraction $\theta (0 < \theta < 1)$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n,$$

where $R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a + \theta h)$ is called the Cauchy's form of Remainder.

Theorem 3.4. (Taylor's Theorem with Lagranges Form of Remainder) If a real valued function defined on $[a, b]$ or $[a, a + h]$ where $a + h = b$ be such that

i) f^{n-1} is continuous on $[a, a + h]$,

ii) f^n exists on $(a, a + h)$,

then, there exists a positive proper fraction $\theta (0 < \theta < 1)$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n,$$

where $R_n = \frac{h^n}{n!}f^n(a + \theta h)$ is called the Lagranges form of Remainder.

Theorem 3.5. (Maclaurin's Theorem) Let a function $f : [0, x] \rightarrow \mathbb{R}$ be such that

i) f^{n-1} is continuous on $[0, x]$,

ii) f^n exists on $(0, x)$,

then there exists a positive proper fraction $\theta (0 < \theta < 1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(\theta h),$$

where p is a positive integer $\leq n$.

$$\text{Hence, } f(x) = f(0) + \sum_{r=1}^{n-1} \frac{x^r}{r!}f^{(r)}(0) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(\theta h).$$

Example 3.5. Expand the function $f(x) = e^x$ about the point $x = 0$ via Taylor's theorem with Lagranges form of remainder.

Sol. Given $f(x) = e^x \Rightarrow f'(x) = e^x = f''(x) = f'''(x) = \cdots$

By Taylor's theorem, with Lagranges form of remainder,

$$\begin{aligned} e^x &= f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x), \quad 0 < \theta < 1 \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}, \quad 0 < \theta < 1 \end{aligned}$$

Expansion of a function as an infinite series. A function $f(x)$ which is defined at $x = a$ and possesses derivative upto n th order at $x = a$, then it can be expressed as an infinite series of the form

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots,$$

if R_n , the remainder of any form, after n terms resulting from Taylor's expansion of $f(x)$ about $x = a$ tends to 0 as $n \rightarrow \infty$, i.e, $\lim_{n \rightarrow \infty} R_n = 0$.

If $a = 0$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{(n)!}f^n(0) + \cdots$$

is called the expansion of $f(x)$ is Maclaurin's infinite series or expansion of $f(x)$ about $x = 0$.

Example 3.6. Expand e^x as an infinite series.

Sol. Let $f(x) = e^x$, then $f^n(x) = e^x$. From Maclaurin's theorem R_n , the remainder after n terms in Lagrange's form is $\frac{x^n}{n!}e$, $0 < \theta < 1$.

Note that $0 < \theta < 1 \Rightarrow 0 < \theta x < x \Rightarrow 1 < e^{\theta x} < e^x$. Also for all real x , $e^{\theta x}$ is bounded.

Let $u_n = \frac{x^n}{n!}$, $u_{n+1} = \frac{x^{n+1}}{(n+1)!}$. Then

$$\begin{aligned} \left| \frac{u_{n+1}}{u_n} \right| &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} \rightarrow 0 \\ &\Rightarrow \lim_{n \rightarrow \infty} u_n \rightarrow 0 \\ &\Rightarrow \lim_{n \rightarrow \infty} R_n \rightarrow 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) = e^x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned}$$